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1972 J. Phys. A: Gen. Phys. 5 L85

(http://iopscience.iop.org/0022-3689/5/9/002)

J. Phys. A: Gen. Phys., Vol. 5, September 1972. Printed in Great Britain. © 1972.

LETTER TO THE EDITOR

Motion of a relativistic electron in a spatially modulated magnetic field

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MS received 29 June 1972

Abstract. The equations of motion of a relativistic electron in a spatially modulated magnetic field are solved for the two limiting cases of a strong or a weak field. The microwave frequency spectrum emitted in the strong field limit depends on a critical field parameter B_{cr}, in qualitative agreement with some recent experimental results of Friedman and Herndon.

Recently Friedman and Herndon (1972) obtained microwaves by passing a pulsed annular electron beam of current 20 A and voltage 720 kV down a waveguide which was surrounded by alternate aluminium and iron rings inside a solenoid. The latter arrangement produces a rippled magnetic field along the wave guide axis. They obtained microwave radiation of wavelength approximately equal to the magnitude of the pitch of the modulating magnetic field when highly relativistic electrons were used. They also found that the frequency spectrum depends on a critical field parameter B_{cr}. We solve the equations of motion in an attempt to give a simple theoretical explanation investigation of the effect.

The equations of motion of a relativistic electron of rest mass m_0 moving with a velocity v along the z axis in a magnetic field B_z are simplest in a frame of reference moving with the electron. In this frame they are

$$m_0\ddot{x}' = \frac{e}{c}\dot{y}'B_z' \tag{1a}$$

$$m_0\ddot{y}' = -\frac{e}{c}\dot{x}'B_z' \tag{1b}$$

$$m_0\ddot{z}' = 0. \tag{1c}$$

The primes label the moving frame and x', y', z' represent the cartesian coordinates of the electron in this frame. The magnetic field in the z direction only is written as

$$B_z' = B^*\left(1 + h(\rho)\cos\frac{2\pi\gamma}{L}(z' - vt')\right). \tag{2}$$

Equation (2) represents a rippled field superimposed upon a DC field. The function $h(\rho)$ is assumed to be a slowly varying function of ρ, and L is the pitch or wavelength of the ripple. Also $\beta = v/c$ and $\gamma^2 = (1-\beta^2)^{-1}$. For simplicity we choose the electron to be at $z' = 0$. Eliminating y' from equations (1*a*) and (1*b*) and dropping the primes we obtain

$$\dddot{x} = -\left(\frac{e}{m_0 c}B_z\right)^2 \dot{x} + \ddot{x}\frac{\dot{B}_z}{B_z}. \tag{3}$$

The solution of (3) depends on the magnitude of the ratio of the second and third terms. We show later that for the values of B^*, ρ, L and h in Friedman and Herndon's experiment, the last term in (3) is small. In this case, the strong field case, (3) becomes

$$\ddot{u} + \left(\frac{eB^*}{m_0 c}\right)^2 (1 + h\cos\mu' t)^2 u = 0 \tag{4}$$

where $u = \dot{x}$ and $\mu' = 2\pi\gamma v/L$. For small h and putting $2\theta = \mu' t$ then (4) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \left(\frac{2eB^*}{\mu' m_0 c}\right)^2 \left\{1 + 16\left(\frac{h}{8}\right)\cos 2\theta\right\} u = 0 \tag{5}$$

which is the standard form of Mathieu's equation. The solution of (5) can be written in many ways but we find the Lindemann–Stieltjes' theory of this equation the most convenient (see Whittaker and Watson 1962). The power series solution of (5) is

$$u = \sum_{n=0}^{\infty} (a_n \xi^n + b_n \xi^{n+1/2}) \tag{6}$$

where $\xi = \cos^2\theta$ and

$$a_1 = 2(2h-1)\left(\frac{eB^*}{\mu' m_0 c}\right)^2 a_0 \tag{7a}$$

$$a_2 = \tfrac{1}{12}\left[\left\{4 + (2h-1)\left(\frac{eB^*}{\mu' m_0 c}\right)^2 a_0\right\} a_1 - 16h\left(\frac{eB^*}{m_0 c\mu'}\right)^2 a_0\right] \tag{7b}$$

$$b_1 = \tfrac{1}{6}\left\{1 + (2h-1)\left(\frac{2eB^*}{\mu' m_0 c}\right)^2\right\} b_0. \tag{7c}$$

Equations (7) are the first few terms of the coefficients a_n and b_n in (6) and where a_0 and b_0 are arbitrary constants determined by the initial conditions. On expansion, (6) becomes

$$\begin{aligned} u = (a_0 + \tfrac{1}{2}a_1 + \tfrac{3}{8}a_2) + \tfrac{1}{2}(a_1 + a_2)\cos\mu' t + \tfrac{1}{8}a_2 \cos 2\mu' t \ldots \\ + b_0 \cos\tfrac{1}{2}\mu' t + b_1 \cos^3(\tfrac{1}{2}\mu' t) \ldots . \end{aligned} \tag{8}$$

Equation (8) expresses the velocity of the electron as a series of cosines of frequency μ' and other harmonics and subharmonics. The coefficient of each term is in fact a power series itself, but for sufficiently small h the series converge. However, each coefficient depends on $eB^*/\mu' m_0 c$ and there exists a critical field parameter B_{cr} defined by $B_{\mathrm{cr}} = (2\pi\gamma m_0 c^2/eL)\beta$ such that there are three regions of the field parameter B^*; $B^* < B_{\mathrm{cr}}$, $B^* \sim B_{\mathrm{cr}}$, $B^* > B_{\mathrm{cr}}$ which make B^*/B_{cr} less than, of the order of, and greater than unity respectively. For $\beta \lesssim 1$ this theoretical value of B_{cr} agrees with the work of Slabospitskii *et al* (1963) for the nonadiabatic change of the motion of an electron in a rippled field.

We now turn to the weak field limit; substituting the series solution of (8) term by term into (3), the ratio of the $\ddot{x}$ term to the $\dot{x}$ term is $h(B_{cr}/B^*)^2$. In Friedman and Herndon's experiment, $h \sim 0{\cdot}05$ and $B_{cr} \sim 6$ kG so consequently the value of B^* necessary (approximately 1 kG) for this term to effectively contribute, is too low to be in the experimental range (3 to 10 kG) of Friedman and Herndon's measurements. Hence we may reasonably neglect the $\ddot{x}$ term in (3).

We now examine the frequency spectrum of the radiation emitted by the accelerated electrons. The intensity spectrum of the fundamental, which also has the largest amplitude, in equation (8) is proportional to

$$\frac{\omega^2}{(\omega-\mu')^2}\sin^2(\omega-\mu')\tau_0 \tag{9}$$

where $2\tau_0$ is the time the electron spends in the rippled field. This spectrum is peaked very strongly at $\omega = \mu'$. For an electron pulse of infinite duration, the broadening of the frequency spectrum in (9) is negligible, but the pulse duration in Friedman and Herndon's experiment was about 50 ns. When $B^* > B_{cr}$ the broadening is less than the case $B^* \sim B_{cr}$ as the peak in the intensity spectrum is sharper. However, the finite lifetime of each pulse, together with small fluctuations in input energy, widen the spectrum such that $\Delta\omega/\omega \sim 0{\cdot}5\,\%$. However, for the case $B^* < B_{cr}$ the spectrum is even broader by a factor of approximately 16. The reason for this is that the intensity of radiation is proportional to approximately $a_1{}^2$ and $a_1{}^2$ is proportional to $(B^*/B_{cr})^2$. Hence for $B^*/B_{cr} \sim \frac{1}{2}$ we have a much broader spectrum: $\Delta\omega/\omega \sim 8\,\%$. These estimates of the widths of the emitted lines are confirmed by Friedman andHerndon's measurements. We have been working in the reference frame of the electron until now and the major frequency radiated is μ'. In the laboratory frame the frequency of the radiation is μ where $\mu = 2\pi v/L$, therefore, $\lambda = L/\beta$. However, $\beta \simeq 0{\cdot}92$ in Friedman and Herndon's experiment so the wavelength of the emitted radiation for highly relativistic electrons is very close to the magnitude of the pitch of the rippled magnetic field. Friedman and Herndon obtained these results experimentally and hence our conclusions about the wavelength and broadening of the radiation are confirmed. Note also that the velocity dependence and independence of magnetic field of the wavelength of the emitted microwaves as shown above has also been confirmed experimentally by the necessary use of relativistic electrons. We also note that the other harmonics of the frequency μ are radiated but are much weaker in intensity. However they are strongest in the region $B^* > B_{cr}$ which may account for the extra lines seen in the spectrum in this region. Finally, we are still trying to understand the amplification mechanism of the radiation in the DC field region and hope to report on this later.

I am most grateful to Dr J C Eilbeck for valuable discussions.

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